Degree of Equiripple Narrow Bandpass FIR Filter

Pavel Zahradnik, Michal Šusta, Miroslav Vlček, and Boris Šimák

***Abstract*—This brief presents the degree equation of an equiripple narrow bandpass finite-impulse response filter, which follows from the filter approximation rather than from empirical data. The degree equation relates the filter specification and the corresponding filter degree. It completes the closed-form filter design, and it is useful in numerical design procedures as well. Two examples demonstrate its functionality and usefulness.**

***Index Terms*—Bandpass filter, degree equation, equiripple (ER) approximation, finite-impulse response (FIR) filter, narrow filter.**

I. Introduction

A reliable degree equation is a welcomed tool in a filter design. In fact, there are two essential concerns in the filter design. First, we wish to obtain the least filter length satisfying the filter specification. Second, if the filter length is acceptable, we wish to evaluate the coefficients of the filter. In the design of optimal (in the Chebyshev sense) equiripple (ER) finite-impulse response (FIR) filters, the degree equation is not an established tool. This is because the dominant method in designing ER FIR filters is the famous Parks–McClellan procedure that is based on numerical optimization. It is represented, e.g., by the m-function *firpm* in MATLAB. The filter length is not known prior to the evaluation of the impulse response of the filter in the Parks–McClellan procedure. The filter length is in fact the result of an iterative evaluation of the impulse response. There are some available approximative degree formulas for ER FIR filters resulting from the fitting of empirical data, e.g., see [3]–[5], where the work in [4] presented the most advanced results. On the other hand, in the closed-form design of ER FIR filters, the filter degree follows from the solution of an approximation problem, and the degree equation is based on the approximating function that is a unique function for each particular filter type. There are exact degree formulas available for some specific ER FIR filters, i.e., for an ER notch FIR filter [7], an ER DC notch FIR filter [8], and an ER comb FIR filter [9], [10]. This brief does not deal with the filter design. It is focused on the degree equation of an ER narrow bandpass (ERNBP) FIR filter only.

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P. Zahradnik, M. Šusta, and B. Šimák are with the Department of Telecommunication Engineering, Faculty of Electrical Engineering, Czech Technical University in Prague, 166 27 Prague, Czech Republic (e-mail: zahradni@fel.cvut.cz; sustamic@fel.cvut.cz; simak@fel.cvut.cz).

M. Vlček is with the Department of Applied Mathematics, Faculty of Transportation Sciences, Czech Technical University in Prague, 110 00 Prague, Czech Republic (e-mail: vlcek@fd.cvut.cz).

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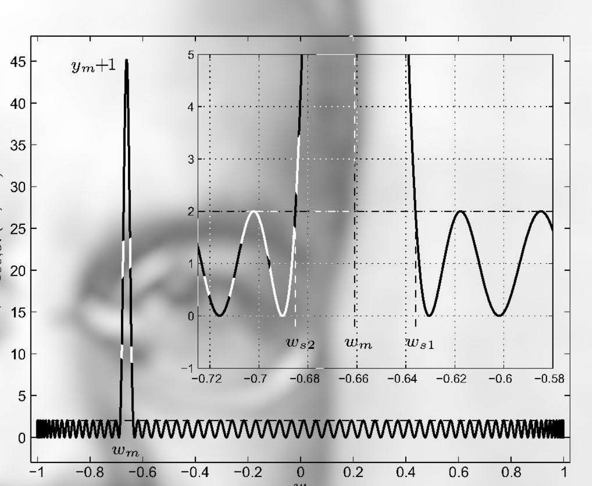

Fig. 1. Polynomial $1 + Z_{100,37}(w, 0.4)$, $w_m = -0.6613$, $y_m = 44.1418$, and a detailed view of the edges of its stopband.

II. Approximating Function of ERNBP FIR Filter

Here, we assume the FIR filters of the most general filter type I, i.e., FIR filters with an odd length of $N = 2n+1$ coefficients and with evenly symmetric impulse response $h(k) = h(N-k)$. We use here the standard notation for elliptic functions [1]. The generating function of an ERNBP FIR filter is in fact identical with that for an ER notch FIR filter [7], i.e., the polynomial $Z_{p,q}(w,\varkappa)$ of degree $n = p + q$ (see Fig. 1). Polynomial $Z_{p,q}(w,\varkappa)$ is defined in terms of Jacobi eta function $H(\phi,\varkappa)$ (e.g., see [7] and Table I), i.e.,

$$Z_{p,q}(w,\varkappa) = \frac{(-1)^p}{2}\left[\left(\frac{H(u+u_0,\varkappa)}{H(u-u_0,\varkappa)}\right)^n + \left(\frac{H(u+u_0,\varkappa)}{H(u-u_0,\varkappa)}\right)^{-n}\right] \qquad (1)$$

where $w = cos(\omega T)$, $\varkappa$ stands for the elliptic modulus, $\mathbf{K}(\varkappa)$ is the Jacobi complete elliptic integral of the first kind (see Table I), and value u_0 is its fraction, i.e.,

$$u_0 = \frac{p}{n}\mathbf{K}(\varkappa) = p_n\mathbf{K}(\varkappa). \qquad (2)$$

Notice that both arguments u and w in (1) are interrelated by the conformal transformation (e.g., see [2, eq. 35]) as

$$w = \frac{\mathrm{sn}^2(u,\varkappa)\mathrm{cn}^2(u_0,\varkappa) + \mathrm{cn}^2(u,\varkappa)\mathrm{sn}^2(u_0,\varkappa)}{\mathrm{sn}^2(u,\varkappa) - \mathrm{sn}^2(u_0,\varkappa)} \qquad (3)$$

which maps the real interval $-1 \le w \le 1$ onto a rectangular trajectory in the complex domain u. Furthermore, by taking into

TABLE I
Jacobi Elliptic Integrals Used Here and the Zeta and Eta Functions

complete, 1st kind	$\mathbf{K}(\varkappa) = \int_0^{\frac{\pi}{2}} \frac{d\Theta}{\sqrt{1-\varkappa^2\sin^2\Theta}}$
incomplete, 1st kind	$\mathbf{F}(\phi,\varkappa) = \int_0^{\phi} \frac{d\Theta}{\sqrt{1-\varkappa^2\sin^2\Theta}}$
complete, 2nd kind	$\mathbf{E}(\varkappa) = \int_0^{\frac{\pi}{2}} \sqrt{1-\varkappa^2\sin^2\Theta}\, d\Theta$
incomplete, 2nd kind	$\mathbf{E}(\phi,\varkappa) = \int_0^{\phi} \sqrt{1-\varkappa^2\sin^2\Theta}\, d\Theta$
incomplete, 3rd kind	$\mathbf{\Pi}(x,\phi,\varkappa) = \int_0^{\phi} \frac{d\Theta}{(1-x\sin^2\Theta)\sqrt{1-\varkappa^2\sin^2\Theta}}$
Zeta	$\mathrm{Z}(\phi,\varkappa) = \mathbf{E}(\phi,\varkappa) - \frac{\mathbf{E}(\varkappa)\mathbf{F}(\phi,\varkappa)}{\mathbf{K}(\varkappa)}$
Eta	$\mathrm{H}(\phi,\varkappa) = \vartheta_1\left(\frac{\pi\phi}{2\,\mathbf{K}(\varkappa)},\varkappa\right) = 2e^{-\pi\frac{\mathbf{K}(\sqrt{1-\varkappa^2})}{4\mathbf{K}(\varkappa)}}\sum_{k=0}^{\infty}(-1)^k e^{-\pi\frac{k(k+1)\mathbf{K}(\sqrt{1-\varkappa^2})}{\mathbf{K}(\varkappa)}}\sin\frac{\pi(2k+1)}{2\mathbf{K}(\varkappa)}\phi$

account the relation between the attenuation in the stopband and the value of the main lobe $a_s = 2/(y_m+1)$ (see Fig. 2), i.e.,

$$y_m = 2\cdot 10^{-a_{sdB}/20} - 1 \qquad (4)$$

we can express the approximating polynomial of an ERNBP FIR filter in the form of a normalized generating polynomial as

$$Q(w) = \frac{Z_{p,q}(w,\varkappa)+1}{y_m+1} = \frac{Z_{p,q}(w,\varkappa)+1}{2\cdot 10^{-a_{sdB}/20}}. \qquad (5)$$

For illustration, see Fig. 2. For further considerations, an explicit representation of the approximating polynomial (1) of real argument w is advantageous because variable u is a complex variable. By the inversion of (3), we get a new nontrivial result, i.e.,

$$u = j\,\mathrm{arcsn}\left(\mathrm{sc}(u_0,\varkappa)\sqrt{\frac{1+w}{1-w}}, \sqrt{1-\varkappa^2}\right). \qquad (6)$$

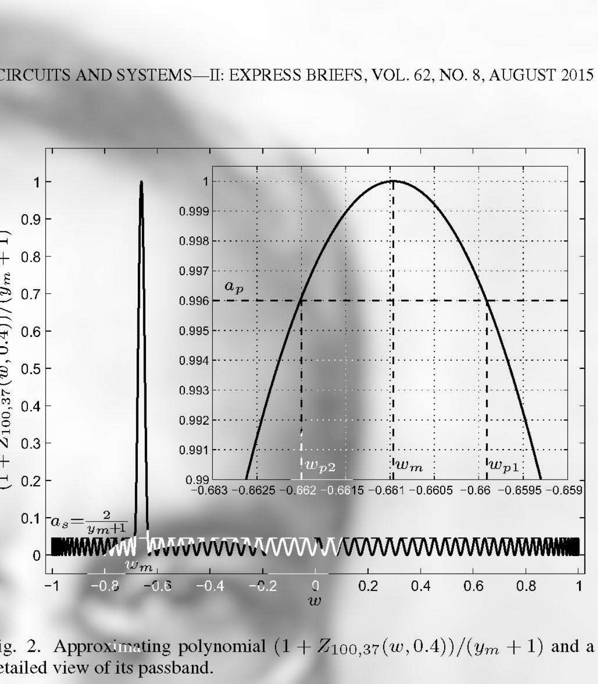

Fig. 2. Approximating polynomial $(1 + Z_{100,37}(w, 0.4))/(y_m + 1)$ and a detailed view of its passband.

Furthermore, it is useful to express generating polynomial (1) in a more compact form, i.e.,

$$Z_{p,q}(w,\varkappa) = (-1)^p\frac{e^{\ln\left[\left(\frac{H(u+u_0,\varkappa)}{H(u-u_0,\varkappa)}\right)^n\right]} + e^{\ln\left[\left(\frac{H(u+u_0,\varkappa)}{H(u-u_0,\varkappa)}\right)^{-n}\right]}}{2}$$
$$= (-1)^p\cosh\left\{n\cdot\ln\left[\frac{H(u+u_0,\varkappa)}{H(u-u_0,\varkappa)}\right]\right\}. \qquad (7)$$

From (6) and (7), we get generating polynomial $Z_{p,q}(w,\varkappa)$ and approximating polynomial $Q(w)$ in real variable w in (8) and (9), respectively, shown at the bottom of the page. Formulas (6), (8), and (9) represent a major achievement. Indeed, they form the basis for the derivation of the degree equation. Auxiliary value σ_m in (9) is

$$\sigma_m = \mathbf{F}\left(\arcsin\frac{\frac{1}{\sqrt{1+\mathrm{cs}(p_n\mathbf{K}(\varkappa),\varkappa)\frac{dn(p_n\mathbf{K}(\varkappa),\varkappa)}{Z(p_n\mathbf{K}(\varkappa),\varkappa)}}}}{\varkappa\,\mathrm{sn}(p_n\mathbf{K}(\varkappa),\varkappa)},\varkappa\right) \qquad (10)$$

where $Z(\phi,\varkappa)$ stands for the Jacobi zeta function, and $\mathbf{\Pi}(x,\phi,\varkappa)$ stands for the Jacobi incomplete elliptic integral of the third kind (see Table I).

$$Z_{p,q}(w,\varkappa) = (-1)^p\cosh\left[n\ln\frac{H\left(j\,\mathrm{arcsn}\left(\mathrm{sc}(p_n\mathbf{K}(\varkappa),\varkappa)\sqrt{\frac{1+w}{1-w}},\sqrt{1-\varkappa^2}\right)+p_n\mathbf{K}(\varkappa),\varkappa\right)}{H\left(j\,\mathrm{arcsn}\left(\mathrm{sc}(p_n\mathbf{K}(\varkappa),\varkappa)\sqrt{\frac{1+w}{1-w}},\sqrt{1-\varkappa^2}\right)-p_n\mathbf{K}(\varkappa),\varkappa\right)}\right] \qquad (8)$$

$$Q(w) = \frac{(-1)^p\cosh\left[n\ln\frac{H\left(j\,\mathrm{arcsn}\left(\mathrm{sc}(p_n\mathbf{K}(\varkappa),\varkappa)\sqrt{\frac{1+w}{1-w}},\sqrt{1-\varkappa^2}\right)+p_n\mathbf{K}(\varkappa),\varkappa\right)}{H\left(j\,\mathrm{arcsn}\left(\mathrm{sc}(p_n\mathbf{K}(\varkappa),\varkappa)\sqrt{\frac{1+w}{1-w}},\sqrt{1-\varkappa^2}\right)-p_n\mathbf{K}(\varkappa),\varkappa\right)}\right]+1}{\cosh 2n\left[(\sigma_m Z(p_n\mathbf{K}(\varkappa),\varkappa) - \mathbf{\Pi}(\sigma_m,p_n\mathbf{K}(\varkappa),\varkappa)\right]+1} \qquad (9)$$

III. Objective of Degree Equation

An ERNBP FIR filter is specified by passband frequency $\omega_m T = \mathrm{acos}(w_m)$; furthermore, by the attenuation in the passband, $a_{pdB} = 20\log(a_p)$ related to the width of the passband $\Delta\omega_p T = (\omega_{p2} - \omega_{p1})T = \mathrm{acos}(w_{p2}) - \mathrm{acos}(w_{p1})$, and finally, by the attenuation in the stopband, $a_{sdB} = 20\log(a_s)$ related to the width of the stopband $\Delta\omega_s T = (\omega_{s2} - \omega_{s1})T = \mathrm{acos}(w_{s2}) - \mathrm{acos}(w_{s1})$ (see Figs. 1 and 2). The best frequency selectivity is achieved for $\Delta\omega_p T \to \Delta\omega_s T$. The aim of the degree equation is to evaluate the least possible filter degree n that satisfies its passband specification $a_{pdB_{act}} \ge a_{pdB}$ in passband $\omega_{p1}T \le \omega T \le \omega_{p2}T$ and the stopband specification $a_{sdB_{act}} \le a_{sdB}$ in stopbands $\omega T \le \omega_{s1}T$ and $\omega T \ge \omega_{s2}T$. In the ERNBP FIR filter, the passband is formed by the main lobe of the generating polynomial, whereas the remaining ripples of equal value form both stopbands. The width and height of the main lobe solely depend on integer value n and on real value $\varkappa$ (1). For a specific value $\varkappa$ and an increasing n, the width of the main lobe decreases, and its height increases, i.e., values $\Delta\omega_p T$, $\Delta\omega_s T$, and a_{sdB} decrease. On the other hand, for a specific value n and an increasing $\varkappa$, the width and height of the main lobe increase, i.e., $\Delta\omega_p T$ and $\Delta\omega_s T$ increase, and a_{sdB} decreases. Hence, both values $\Delta\omega_p T$ and $\Delta\omega_s T$ exhibit an identical trend when varying values n and $\varkappa$. Consequently, in general, it is not possible to meet passband specification a_{pdB} for $\Delta\omega_p T$ and stopband specification a_{sdB} for $\Delta\omega_s T$ at the same time, particularly not if approaching the ideal filter with $\Delta\omega_p T \to \Delta\omega_s T$.

IV. Filter Specification and Degree

By taking into account that the approximating polynomial is at the passband edges, $Q(w_{p1}) = Q(w_{p2}) = 10^{a_{pdB}/20}$, which is apparent in Fig. 2, we can express (11) and (12), shown at the bottom of the page, and from them we get (13), also shown at the bottom of the page. Furthermore, we require value $H_{w_{p1}}$ in (13) to be real. From (13) and from relation $\Im\{H_{w_{p1}}\} = 0$, we obtain the real values $\varkappa$ and w_{p1} that are always within limits $0 < \varkappa < 1$ and $w_m < w_{p1} < w_{s1}$.

Finally, from (11), we express the degree equation of the ERNBP FIR filter in the form

$$n \ge \frac{\mathrm{acosh}\left(2.10^{0.05(a_{pdB}-a_{sdB})}-1\right)}{\frac{H\left(j\,\mathrm{arcsn}\left(\mathrm{sc}(p_n\mathbf{K}(\varkappa),\varkappa)\sqrt{\frac{1+w_{p1}}{1-w_{p1}}},\sqrt{1-\varkappa^2}\right)+p_n\mathbf{K}(\varkappa),\varkappa\right)}{H\left(j\,\mathrm{arcsn}\left(\mathrm{sc}(p_n\mathbf{K}(\varkappa),\varkappa)\sqrt{\frac{1+w_{p1}}{1-w_{p1}}},\sqrt{1-\varkappa^2}\right)-p_n\mathbf{K}(\varkappa),\varkappa\right)}}. \qquad (14)$$

Notice that, for degree n in (14), passband specification a_{pdB} for $\Delta\omega_p T$ and stopband attenuation a_{sdB} are always met. On the other hand, depending on the filter specification, the specified width of the stopband $\Delta\omega_s T$, in general, cannot be simultaneously guaranteed for any filter degree n with the passband specification (see the end of Section III). We can evaluate the actual width $\Delta\omega_{s\,act}T$ using (9). If the passband and stopband specifications cannot be simultaneously met and if we prefer to meet just the stopband specification, we can alternatively evaluate degree n_s related to the stopband specification (see our first example). Of course, each degree n_s cannot meet the passband specification. Value n_s follows from generating polynomial (1), and it is related to the degree of the ER notch FIR filter, cf., [7, eq. 35], i.e.,

$$n_s \ge \frac{\frac{1}{2}\ln\left(2.10^{-a_{sdB}/20}-1+\sqrt{(2.10^{-a_{sdB}/20}-1)^2-1}\right)}{\sigma_{m_s}Z(p_n\mathbf{K}(\varkappa_s),\varkappa_s) - \mathbf{\Pi}(\sigma_{m_s},p_n\mathbf{K}(\varkappa_s),\varkappa_s)}. \qquad (15)$$

Auxiliary value σ_{m_s} in (15) is

$$\sigma_{m_s} = \mathbf{F}\left(\arcsin\frac{\sqrt{\frac{\cos(\omega_m T)-\cos\left[(\omega_m+\frac{1}{2}\Delta\omega_s)T\right]}{\cos(\omega_m T)+1}}}{\varkappa_s\,\mathrm{sn}(p_n\mathbf{K}(\varkappa_s),\varkappa_s)},\varkappa_s\right). \qquad (16)$$

Elliptic modulus $\varkappa_s$ [7] can be advantageously expressed in a simpler form as follows:

$$\varkappa_s = \sqrt{\frac{2\left(\cos(\omega_m - \frac{1}{2}\Delta\omega_s)T - \cos(\omega_m + \frac{1}{2}\Delta\omega_s)T\right)}{\left(1+\cos(\omega_m-\frac{1}{2}\Delta\omega_s)T\right)\left(1-\cos(\omega_m+\frac{1}{2}\Delta\omega_s)T\right)}}. \qquad (17)$$

Formula (17) is new as well.

$$\frac{(-1)^p\cosh\left[n\ln\frac{H\left(j\,\mathrm{arcsn}\left(\mathrm{sc}(p_n\mathbf{K}(\varkappa),\varkappa)\sqrt{\frac{1+w_{p1}}{1-w_{p1}}},\sqrt{1-\varkappa^2}\right)+p_n\mathbf{K}(\varkappa),\varkappa\right)}{H\left(j\,\mathrm{arcsn}\left(\mathrm{sc}(p_n\mathbf{K}(\varkappa),\varkappa)\sqrt{\frac{1+w_{p1}}{1-w_{p1}}},\sqrt{1-\varkappa^2}\right)-p_n\mathbf{K}(\varkappa),\varkappa\right)}\right]+1}{2.10^{-a_{sdB}/20}} - 10^{a_{pdB}/20} = 0 \qquad (11)$$

$$\frac{(-1)^p\cosh\left[n\ln\frac{H\left(j\,\mathrm{arcsn}\left(\mathrm{sc}(p_n\mathbf{K}(\varkappa),\varkappa)\sqrt{\frac{1+\cos(\arccos(w_{p1})+\Delta\omega_pT)}{1-\cos(\arccos(w_{p1})+\Delta\omega_pT)}},\sqrt{1-\varkappa^2}\right)+p_n\mathbf{K}(\varkappa),\varkappa\right)}{H\left(j\,\mathrm{arcsn}\left(\mathrm{sc}(p_n\mathbf{K}(\varkappa),\varkappa)\sqrt{\frac{1+\cos(\arccos(w_{p1})+\Delta\omega_pT)}{1-\cos(\arccos(w_{p1})+\Delta\omega_pT)}},\sqrt{1-\varkappa^2}\right)-p_n\mathbf{K}(\varkappa),\varkappa\right)}\right]+1}{2.10^{-a_{sdB}/20}} - 10^{a_{pdB}/20} = 0 \qquad (12)$$

$$\frac{H\left(j\,\mathrm{arcsn}\left(\mathrm{sc}(p_n\mathbf{K}(\varkappa),\varkappa)\sqrt{\frac{1+w_{p1}}{1-w_{p1}}},\sqrt{1-\varkappa^2}\right)+p_n\mathbf{K}(\varkappa),\varkappa\right)}{H\left(j\,\mathrm{arcsn}\left(\mathrm{sc}(p_n\mathbf{K}(\varkappa),\varkappa)\sqrt{\frac{1+w_{p1}}{1-w_{p1}}},\sqrt{1-\varkappa^2}\right)-p_n\mathbf{K}(\varkappa),\varkappa\right)} - \frac{H\left(j\,\mathrm{arcsn}\left(\mathrm{sc}(p_n\mathbf{K}(\varkappa),\varkappa)\sqrt{\frac{1+\cos(\arccos(w_{p1})+\Delta\omega_pT)}{1-\cos(\arccos(w_{p1})+\Delta\omega_pT)}},\sqrt{1-\varkappa^2}\right)+p_n\mathbf{K}(\varkappa),\varkappa\right)}{H\left(j\,\mathrm{arcsn}\left(\mathrm{sc}(p_n\mathbf{K}(\varkappa),\varkappa)\sqrt{\frac{1+\cos(\arccos(w_{p1})+\Delta\omega_pT)}{1-\cos(\arccos(w_{p1})+\Delta\omega_pT)}},\sqrt{1-\varkappa^2}\right)-p_n\mathbf{K}(\varkappa),\varkappa\right)} = H_{w_{p1}} - H_{w_{p2}} = 0 \qquad (13)$$

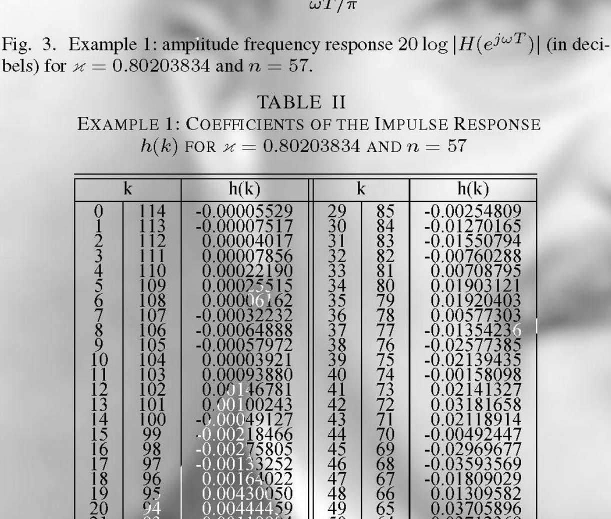

Fig. 3. Example 1: amplitude frequency response $20\log|H(e^{j\omega T})|$ (in decibels) for $\varkappa = 0.80203834$ and $n = 57$.

TABLE II
Example 1: Coefficients of the Impulse Response $h(k)$ for $\varkappa = 0.80203834$ and $n = 57$

k		h(k)	k		h(k)
0	114	-0.00005529	29	85	-0.00254809
1	113	-0.00007517	30	84	-0.01270165
2	112	0.00004017	31	83	-0.01550794
3	111	0.00007856	32	82	-0.00760288
4	110	0.00022190	33	81	0.00708795
5	109	0.00025515	34	80	0.01903121
6	108	0.00006162	35	79	0.01920403
7	107	-0.00032232	36	78	0.00577303
8	106	-0.00064888	37	77	-0.01354236
9	105	-0.00057972	38	76	-0.02577385
10	104	0.00003921	39	75	-0.02139435
11	103	0.00093880	40	74	-0.00158098
12	102	0.00146781	41	73	0.02141327
13	101	0.00100243	42	72	0.03181658
14	100	-0.00049127	43	71	0.02118914
15	99	-0.00218466	44	70	-0.00492447
16	98	-0.00275805	45	69	-0.02969677
17	97	-0.00133252	46	68	-0.03593569
18	96	0.00164022	47	67	-0.01809029
19	95	0.00430050	48	66	0.01309582
20	94	0.00444459	49	65	0.03705896
21	93	0.00119994	50	64	0.03712369
22	92	-0.00388502	51	63	0.01220849
23	91	-0.00739078	52	62	-0.02176094
24	90	-0.00621738	53	61	-0.04215140
25	89	-0.00008779	54	60	-0.03489773
26	88	0.00754711	55	59	-0.00431425
27	87	0.01129506	56	58	0.02947758
28	86	0.00751387	57		0.04401221

V. Examples

Example 1: In order to demonstrate the functionality of degree equation (14), let us evaluate the length of the ERNBP FIR filter specified by $\omega_m T = 0.27\pi$, by $a_{pdB} = -3$ for $\Delta\omega_p T = 0.03\pi$, and by $a_{sdB} = -80$ for $\Delta\omega_s T = 0.1\pi$.

Based on our results in Section IV, we get $\varkappa = 0.80203834$ (13) and $n = 57$ (14), i.e., the filter length is $N = 115$ coefficients. The amplitude frequency response of the filter is shown in Fig. 3. For the sake of inspection, the impulse response of the filter is summarized in Table II. The actual passband attenuation is $a_{pdB\text{ act}} = -3.0000 = a_{pdB}$ for $\Delta\omega_p T = 0.03\pi$, i.e., the passband specification is met. Stopband attenuation $a_{sdB} = -81.9655 < a_{sdB} = -80$ is met as well (see Fig. 3). Notice that, for the nearest lower degree $n = 56$, the actual attenuation in the stopband is $a_{sdB\text{ act}} = -78.6484 > a_{sdB} = -80$, i.e., the specified value $a_{sdB} = -80$ is no longer met. This fact demonstrates the correctness of degree equation (14). Furthermore, notice that, for the correct filter degree $n = 57$, the actual values in the stopband are $a_{sdB\text{ act}} = -42.2989 > a_{sdB} = -80$ for $\Delta\omega_s T = 0.1\pi$ and $\Delta\omega_{s\text{ act}}T = 0.1192\pi$ for $a_{sdB} = -80$, indicating that the stopband specification cannot be satisfied. For details regarding the partial satisfaction of the filter specification, recall Sections III and IV. In order to meet just the stopband specification, we get $\varkappa_s = 0.75573473$ (17) and $n_s = 67$ (15), i.e., the filter length is $N_s = 135$ coefficients (see Table III). The amplitude frequency response of the filter is shown in Fig. 4. The actual stopband attenuation is $a_{sdB\text{ act}} = -80.0017 < a_{sdB} = -80$ for the specified $\Delta\omega_s T = 0.1\pi$, i.e., the stopband specification is met. The actual passband attenuation is $a_{pdB\text{ act}} = -4.2331 < a_{pdB} = -3$ for the specified width $\Delta\omega_p T = 0.03\pi$, i.e., the passband specification cannot be met. For the sake of comparison, the filter length estimation presented in [4] for the filter specification in this example is $\hat{N}_{BP} = 131$ coefficients. The inaccuracy of the estimation [4] is approximately -3% compared with the exact value.

TABLE III
Example 1: Coefficients of the Impulse Response $h(k)$ for $\varkappa = 0.75573473$ and $n = 67$

k		h(k)	k		h(k)
0	134	0.00005702	34	100	-0.01133275
1	133	0.00006261	35	99	-0.00579335
2	132	0.00002236	36	98	0.00486869
3	131	-0.00007730	37	97	0.01352782
4	130	-0.00017424	38	96	0.01350657
5	129	-0.00016609	39	95	0.00362407
6	128	0.00000673	40	94	-0.01017658
7	127	0.00027507	41	93	-0.01830493
8	126	0.00043900	42	92	-0.01415488
9	125	0.00029357	43	91	0.00067306
10	124	-0.00018018	44	90	0.01661199
11	123	-0.00071211	45	89	0.02226688
12	122	-0.00086243	46	88	0.01254307
13	121	-0.00034486	47	87	-0.00702587
14	120	0.00066052	48	86	-0.02330166
15	119	0.00148842	49	85	-0.02437770
16	118	0.00138603	50	84	-0.00827663
17	117	0.00012298	51	83	0.01484279
18	116	-0.00164305	52	82	0.02907965
19	115	-0.00263476	53	81	0.02378019
20	114	-0.00182357	54	80	0.00149074
21	113	0.00065530	55	79	-0.02305793
22	112	0.00328878	56	78	-0.03274067
23	111	0.00403914	57	77	-0.02006145
24	110	0.00184323	58	76	0.00708570
25	109	-0.00229954	59	75	0.03032749
26	108	-0.00562839	60	74	0.03334799
27	107	-0.00539208	61	73	0.01342850
28	106	-0.00100522	62	72	-0.01623185
29	105	0.00503741	63	71	-0.03533527
30	104	0.00846711	64	70	-0.03050337
31	103	0.00618596	65	69	-0.00472575
32	102	-0.00113440	66	68	0.02448945
33	101	-0.00888309	67		0.03717115

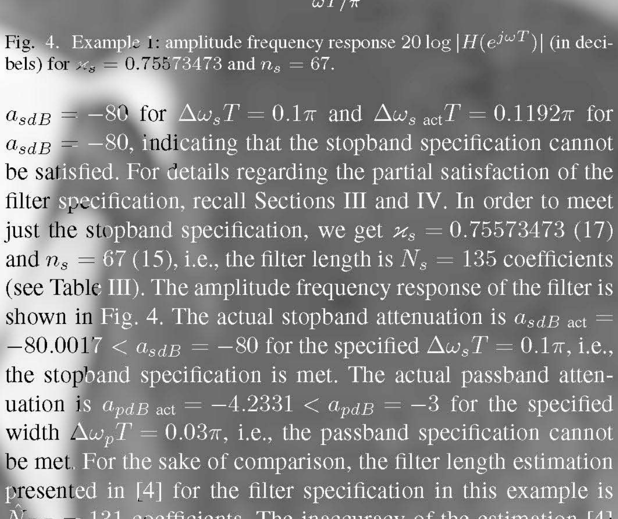

Fig. 4. Example 1: amplitude frequency response $20\log|H(e^{j\omega T})|$ (in decibels) for $\varkappa_s = 0.75573473$ and $n_s = 67$.

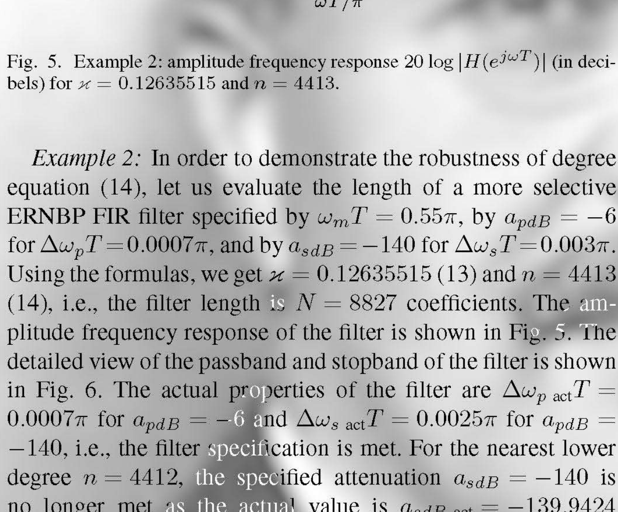

Fig. 5. Example 2: amplitude frequency response $20\log|H(e^{j\omega T})|$ (in decibels) for $\varkappa = 0.12635515$ and $n = 4413$.

Example 2: In order to demonstrate the robustness of degree equation (14), let us evaluate the length of a more selective ERNBP FIR filter specified by $\omega_m T = 0.55\pi$, by $a_{pdB} = -6$ for $\Delta\omega_p T = 0.0007\pi$, and by $a_{sdB} = -140$ for $\Delta\omega_s T = 0.003\pi$. Using the formulas, we get $\varkappa = 0.12635515$ (13) and $n = 4413$ (14), i.e., the filter length is $N = 8827$ coefficients. The amplitude frequency response of the filter is shown in Fig. 5. The detailed view of the passband and stopband of the filter is shown in Fig. 6. The actual properties of the filter are $\Delta\omega_{p\text{ act}}T = 0.0007\pi$ for $a_{pdB} = -6$ and $\Delta\omega_{s\text{ act}}T = 0.0025\pi$ for $a_{pdB} = -140$, i.e., the filter specification is met. For the nearest lower degree $n = 4412$, the specified attenuation $a_{sdB} = -140$ is no longer met as the actual value is $a_{sdB\text{ act}} = -139.9424$ only. This result proves the correctness of degree equation (14). The filter length estimation presented in [4] for the filter specification in this example is $\hat{N}_{BP} = 5637$ coefficients. The inaccuracy of the estimation in [4] is approximately -36% compared with the exact value.

VI. Conclusion

We have developed and presented an exact degree equation for an ERNBP FIR filter. It relates the filter specification and a corresponding filter degree. The ER approximation based on a single lobe that forms the passband of the filter, in general, does not allow to fulfill both the passband and stopband specifications at the same time. The presented degree equation reflects this fact.

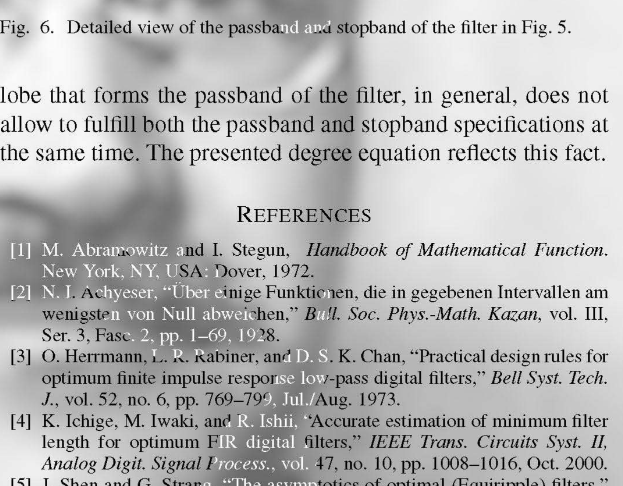

Fig. 6. Detailed view of the passband and stopband of the filter in Fig. 5.

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